Introduction to CGE and SCGE Modelling: Lecture Notes

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General context

- Three broad types of methods in computational economics:
  - Statistical and econometric analysis
  - Computable general (or partial) equilibrium analysis
  - Agent-based modelling (microsimulations)
- Major motivation for the use of all these methods: policy evaluation and forecasting
- The last method is probably most promising for the future, but for now - low practical use for the purposes of policy evaluation or forecasting
- Treatment of model parameters or data in general: stochastic vs. deterministic
- CGE and econometric approach can be merged: e.g. estimation of elasticity parameters

GE vs. CGE

- A CGE model, shortly, is a system of equations describing the behaviour of the economic agents, the structure of the markets and the institutions, and the links between them, one solution to which is believed to be known from the observed data
- As one can suspect, a CGE model must have some relation to a general equilibrium model known from a microeconomics textbook ('a theoretical model filled with data')
- However, the link between the two is not as obvious as it may seem
- To later better understand this link we will start by shortly reviewing the GE theory
- After we have recalled the basic concepts, we will formulate a textbook competitive GE model and see what happens when we try to fill it with data

CGE approach

- The micro textbook exercises on GE theory start with specifying the technology and preferences, markets setup, as well as endowments and profit distribution rules
- In a *CGE approach* you start from a set of data that records economic transactions, and not from a description of an economy in terms of mathematical functions
- The basic problem of modelling: how to make use of the data, such that it will be possible to perform policy analysis and forecasting
- The answer of a CGE approach is: assume that the data describes the equilibrium of some deterministic model, and try to come up with a guess of this model

Consumer behaviour

- A standard microeconomic model describes a toy world inhabited by many rational individuals (consumers)
- These individuals are capable of making consistent decisions about all kinds of issues, and act independently of all others
- Preferences of a given individual c are assumed to be described by a utility function u_c with certain mathematical properties (e.g. strictly increasing)
- Consumers are price-takers, no strategic interaction
- They maximize utility subject to an individual budget constraint
- The result of their optimization can be expressed as a set of demand functions, which completely characterize the preferences

Representation of preferences

- A simple consumer's problem:

$$\max_{\mathbf{d}_c} \quad u_c(\mathbf{d}_c) \quad s.t. \quad \sum_i p_i \cdot d_{ic} \leq M_c,$$

  where $\mathbf{d}_c$ is a vector of consumption quantities, p_i are given product prices, and M_c is income
- Solution is a set of (Marshalian) demand functions:

$$d_{ic}^* = d_{ic}(\mathbf{p}, M_c)$$

- Properties: continuous, homogenous of degree 0
- If preferences are strictly monotonic (we will only have such cases) => budget constraint holds with equality

Representation of preferences

- An alternative (dual) approach:

$$\min_{\mathbf{d}_c} \quad \mathbf{p} \cdot \mathbf{d}_c \quad s.t. \quad u_c(\mathbf{d}_c) \geq u_c^*$$

- The result is an expenditure function, which also completely characterizes the preferences

$$e_c(\mathbf{p}, u_c^*)$$

- Under some additional assumptions (homothetic preferences), it is possible to get the following representation:

$$e_c(\mathbf{p}, u_c^*) = \tilde{e}_c(\mathbf{p}) \cdot u_c^*$$

- Then, the unit expenditure function $\tilde{e}_c(\mathbf{p})$ can be used as a natural price index for the single consumer (we will use it later)

Producer behaviour

- Firms are characterized by technology: way of converting inputs into outputs
- Technology of a firm k is repserented by a production function f_k that has certain mathematical properties (e.g. non-increasing returns to scale)
- Firms have to choose both the amount of inputs and the amount of outputs, in order to maximize profits
- Firms use intermediate goods inputs and factor inputs
- Factors are owned by the households and are in fixed supply
- Firms are price-takers, no strategic interaction (else: non-Walrasian setup)

Representation of technology

- Producer's problem: Step 1 - Cost function

$$c_k(\mathbf{p}, \mathbf{w}, g_k) = \min_{\mathbf{a}_k, \mathbf{b}_k} \quad \mathbf{p} \cdot \mathbf{a}_k + \mathbf{w} \cdot \mathbf{b}_k \quad s.t. \quad f_k(\mathbf{a}_k, \mathbf{b}_k) \geq g_k,$$

  where $\mathbf{a}_k$ is the intermediate inputs vector, $\mathbf{b}_k$ is the factor inputs vector, g_k is the activity level (in units of 'throughput')
- Shephard's lemma gives the (conditional) demand functions

$$a_{ik}^* = a_{ik}(\mathbf{p}, \mathbf{w}, g_k) = \frac{\partial c_k(\mathbf{p}, \mathbf{w}, g_k)}{\partial p_i}$$

$$b_{jk}^* = b_{jk}(\mathbf{p}, \mathbf{w}, g_k) = \frac{\partial c_k(\mathbf{p}, \mathbf{w}, g_k)}{\partial w_j}$$

Representation of technology

- Producer's problem: Step 2 - Revenue function

$$r_k(\mathbf{p}, g_k) = \max_{\mathbf{y}_k} \quad \mathbf{p} \cdot \mathbf{y}_k \quad s.t. \quad h_k(\mathbf{y}_k) \leq z_k(g_k),$$

  where $\mathbf{y}_k$ is the output vector, $h_k(\mathbf{y}_k)$ is a concave input requirement function, $z_k(g_k)$ is the scale economies relationship, and $\mathbf{p}$ is a given price vector
- The outcome is a conditional supply function:

$$y_{ik}^* = y_{ik}(\mathbf{p}, g_k) = \frac{\partial r_k(\mathbf{p}, g_k)}{\partial p_i}$$

- In textbook examples, most often $h_k(\cdot) = y_k$, but in CGE applications, we will often have the multiple output case

Representation of technology

- Producer's problem: Step 3 - Profit maximization
- In the end, firms choose the level of activity in order to maximize the profits

$$\pi_k(\mathbf{p}, \mathbf{w}) = \max_{g_k} \quad r_k(\mathbf{p}, g_k) - c_k(\mathbf{p}, \mathbf{w}, g_k)$$

- The optimality condition is the well known rule $MR = MC$

$$\frac{\partial r_k(\mathbf{p}, g_k^*)}{\partial g_k} = \frac{\partial c_k(\mathbf{p}, \mathbf{w}, g_k^*)}{\partial g_k}$$

- This finishes the description of firms's optimization
- Now, we put two sides of economy together

Consumer income

- The still missing link between the two parts of economy is the specification of income
- Individuals get income by providing factor service to the firms and getting a share of the firms' profits:

$$M_c(\mathbf{p}, \mathbf{w}) = \sum_j w_j E_{cj} + \sum_k \omega_{ck} \pi_k(\mathbf{p}, \mathbf{w}),$$

  where E_{cj} is the endowment of individual c with factor j, ω_{ck} is the share of individual c in the profits of firm k
- Endowments and profit shares are exogenous, fixed parameters

General equilibrium

- Equilibrium in the market system is achieved when the demands of buyers match the supplies of sellers at prevailing prices in every market simultaneously
- We define a real-valued *aggregate excess demand* function for each commodity (product or factor) market:

$$\begin{aligned} z_i^a(\mathbf{p},\mathbf{w}) &= \sum_k a_{ik}(\mathbf{p},\mathbf{w},g_k^*) + \sum_c d_{ic}(\mathbf{p},M_c(\mathbf{p},\mathbf{w})) - \sum_k y_{ik}(\mathbf{p},g_k^*) \\ z_j^b(\mathbf{p},\mathbf{w}) &= \sum_{ik} b_{jk}(\mathbf{p},\mathbf{w},g_k^*) - \sum_c E_{cj} \end{aligned}$$

- And the aggregate excess demand *vector* is then:

$$\mathbf{z}(\mathbf{p},\mathbf{w}) = \left(z_1^a(\cdot),\ \dots,z_I^a(\cdot),\ z_1^b(\cdot),\ \dots,\ z_J^b(\cdot)\right)$$

- Walrasian equilibrium: prices that clear all markets:

$$\mathbf{z}(\mathbf{p}^*,\mathbf{w}^*) = \mathbf{0}$$

Properties of aggregate excess demand function

- 1. Continuous at **p**
- 2. Homogenous of degree 0
- It means that the system $\mathbf{z}(\mathbf{p}^*, \mathbf{w}^*) = \mathbf{0}$ has infinitely many solutions
- We need one more equation to fix the absolute price level
- This equation defines the units of account, or the *numeraire*
- But then we have more equations then unknowns! Need to drop one equation somehow
- Solution: *Walras' law*

Properties of aggregate excess demand function

- 3. Walras' law: the value of aggregate access demand is zero at *any* set of positive prices:
- $\mathbf{p} \cdot \mathbf{z}^a(\mathbf{p}, \mathbf{w}) + \mathbf{w} \cdot \mathbf{z}^b(\mathbf{p}, \mathbf{w}) = 0$
- It is a purely mathematical result that can be derived when all budget constraints of the households hold with equality
- The consequence of this result is that the full system of aggregate excess demands is overidentified
- Thus, one of the equations can be dropped and we end up with an exactly identified system

Complementarity format

- The optimization problems faced by firms and households actually have to be solved subject to an additional set of constraints, namely the non-negativity of prices and quantities
- This was first recognized in the 1930s, and the seminal contributions to the economic theory done by Arrow, Debreu, and McKenzie are all based on a system of weak inequilities, rather than equations. This approach allows for *corner solutions*, when some prices or quantities may be zero
- This syntax is not needed if you have a priori knowledge that no price and no quantity will go to zero. However, for the sake of generality, it is better to formulate the model in the *complementarity* syntax

Complementarity format

- So, what you do is you perform the consumer's optimization subject to non-negativity of demand, and the firm's optimization - subject to non-negativity of activity levels
- Then, you formulate the general equilibrium conditions as:

$$\begin{aligned} \mathbf{z}^a(\mathbf{p}^*, \mathbf{w}^*) &\leq \mathbf{0} \\ \mathbf{z}^b(\mathbf{p}^*, \mathbf{w}^*) &\leq \mathbf{0} \\ \mathbf{p}^* &\geq \mathbf{0} \\ \mathbf{w}^* &\geq \mathbf{0} \\ \mathbf{p}^* . \cdot \mathbf{z}^a(\mathbf{p}^*, \mathbf{w}^*) &= \mathbf{0} \\ \mathbf{w}^* . \cdot \mathbf{z}^b(\mathbf{p}^*, \mathbf{w}^*) &= \mathbf{0} \end{aligned}$$

The complete system

- The complete system has the following form:

$$
\begin{array}{rcllr}
M_c & = & \sum_j w_j E_{cj} + \sum_k \theta_{ck} \pi_k(\mathbf{p}, \mathbf{w}) & & \forall c \\
\frac{\partial r_k(\mathbf{p}, g_k)}{\partial g_k} & \leq & \frac{\partial c_k(\mathbf{p}, \mathbf{w}, g_k)}{\partial g_k} & \perp \quad g_k \geq 0 & \forall k \\
z_i^a(\mathbf{p}, \mathbf{w}) & \leq & 0 & \perp \quad p_i \geq 0 & \forall i \\
z_j^b(\mathbf{p}, \mathbf{w}) & \leq & 0 & \perp \quad w_j \geq 0 & \forall j
\end{array}
$$

- If utility functions are continuos, strongly increasing, and strictly quasiconcave, while the production functions are continuous and convex, the solution to this problem exists

Extending the simple setup

- We will later drop the distinction between goods and factors to save on notation and to allow the consumption of factors (labour-leisure choice)
- State may collect taxes and make transfers, or provide public goods
- Mobility of goods and services (trade) as well as of production factors (intenational capital markets, migration) can be introduced
- More complex extensions: dynamics, uncertainty
- Equilibrium can also be shown to exist if taxes are introduced, or several regions (time periods), but in general not for the cases with increasing returns to scale or market failures

CGE approach

- The micro textbook exercises on GE theory start with specifying the technology and preferences, markets setup, as well as endowments and profit distribution rules
- The major difference between the GE theory and CGE modelling is that in a computational approach you start from a set of data that records economic transactions, and not from a description of an economy in terms of mathematical functions
- The basic problem of modelling is as follows: how to make use of the data, such that it will be possible to perform policy analysis and forecasting
- The answer of a CGE approach is: assume that the data describes the equilibrium of some deterministic model, and try to come up with a guess of this model

The algorithm of CGE modelling

- 1. Formulate a research question and collect raw data
- 2. Define agents, commodities (anything for which there is demand), and institutions
- 3. Organize data in a benchmark equilibrium dataset
- 4. Specify market forms (in case of imperfect competition) and prices (in case of taxes or markups)
- 5. Specify technology and preferences, adopt functional forms
- 6. Specify macroeconomic closure rules (if institutions are present), complete the mathematical formulation

The algorithm of CGE modelling

- 7. Numerical specification (calibration)
- 8. Choosing solution (programming) strategy
- 9. Benchmark replication
- 10. Run experiments
- 11. Evaluate the outcomes
- 12. Sensitivity analysis
- We will now demonstrate the use of the algorithm using a very simple example in GAMS and then we will look at some extensions: *handout on 2x2x1 Model*

Aggregation

- An important distinction between textbook GE and CGE approach is the size of interacting agents
- The problem of connecting the two (how to derive macroeconomic balancing conditions from individual optimization) is in fact the problem of consistent *aggregation*
- A generally used assumption in CGE modelling is that of a *representative household* and a *representative firm*
- The sum of individuals' choices is mathematically equivalent to the decision of one individual, only if their demand functions are linear in expenditure and identical up to the addition of a function that is independent of expenditure (Gorman, 1953)

Aggregation

- For the firms, if the cost function of each individual firm is of generalized linear form, an exact aggregate cost function for the industry exists (Gorman, 1968)
- Even if these two conditions are fulfilled, the existence proofs are available for a rather short list of models and not available for many cases with market imperfections
- Having these issues in mind, we thus can assume functional forms for the aggregate of all households and for the aggregate of firms in the industry
- However, it is important to remember that the claim of micro-foundations of a CGE model is not always valid
- This however does not make CGE models a less useful policy analysis tool

A CES functional form: intro

- A function with *constant elasticity of substitution* is a workhorse of the CGE modelling, and also a victim of constant critisism
- It implies relatively simple expressions for the demand functions and gives some flexibility in terms of pattern of substitution
- It plays an important role in the new economic geography (NEG, Krugman) literature, as a basis for Dixit-Stiglitz framework
- Special cases of CES are the Cobb-Douglas (linear-logarithmic) and Leontief (fixed coefficients) forms

CES: basic forms

- We will consider the following form of the CES utility function

$$U_c = \varphi_c \cdot \left(\sum_i \gamma_{ic}^{1-\rho_c} d_{ic}^{\rho_c} \right)^{1/\rho_c}$$

and of the production function

$$g_k = \psi_k \cdot \left(\sum_j \beta_{jk}^{1-\rho_k} b_{jk}^{\rho_k} \right)^{1/\rho_k}$$

- See the notes in *ces.pdf* to trace the derivation of other CES forms
- Elasticity of substitution $\sigma = \frac{1}{1-\rho}$
- A CET function (e.g. input requirement function $h_k(\mathbf{y}_k)$): just replace σ by $-\theta$, $\theta > 0$

CES: derived forms

- For brevity we drop the destinction between factors and goods: i counts all commodity markets
- Unit cost function (Note: we dropped the tilde above c, e, r!):

$$c_k(\mathbf{p}) = \frac{1}{\psi_k}\left(\sum_i \beta_{ik} p_i^{1-\sigma_k}\right)^{\frac{1}{1-\sigma_k}}$$

- Input demand function:

$$b_{ik}(\mathbf{p}, g_k) = g_k \cdot \beta_{ik} \cdot \psi_k^{\sigma_k - 1}\left(\frac{\tilde{c}_k}{p_i}\right)^{\sigma_k}$$

- Unit expenditure function:

$$e_c(\mathbf{p}) = \frac{1}{\varphi_c}\left(\sum_i \gamma_{ic} p_i^{1-\sigma_c}\right)^{\frac{1}{1-\sigma_c}}$$

- Consumer demand function:

$$d_{ic}(\mathbf{p}, M_c) = \gamma_{ic} \cdot \varphi_c^{\sigma_c - 1}\frac{M_c}{\tilde{e}_c}\left(\frac{\tilde{e}_c}{p_i}\right)^{\sigma_c}$$

CES calibration

- *Calibrated share form* syntax
- We have introduced some shift parameters in the functions that can be removed from the model formulation (thus, we don't have to calibrate them) by applying the following technique:
- 1. Write down all equations evaluated at the initial equlibrium
- 2. Express the shift and share parameters as functions of the initial values of variables
- 3. Substitute the derived expressions for the parameters in the general formulation of the model
- This is not always equivalent to just dividing both parts of equations by their benchmark values!!!

MCM for a larger model

		farms g_1	manuf. g_2	services g_3	Jones M_1	Gates M_2	Σ
p_1	food	10			-8	-2	0
p_2	textiles		50-10	-5	-20	-15	0
p_3	other manuf.	-5	50-20	-5	-10	-10	0
p_4	bus.-serv.	-1	-14	40-25			0
p_5	househ.-serv.			80	-40	-40	0
p_6	labour	-3	-20	-55	100-25	10-7	0
p_7	capital	-1	-36	-30	3	64	0
	Σ	0	0	0	0	0	0

- Included are the industry-commodity framework (multiple output) and labour-leisure choice

Functional forms

- f counts firms, h counts households, i counts commodities
- Constant returns to scale: $z_f(g_f) = g_f$
- Cost and revenue functions are linear in activity level
- $\tilde{\beta}, \tilde{\mu}, \tilde{\gamma}$ are calibrated shares
- CES unit cost function

$$c_f(\mathbf{p}) = \frac{1}{\psi_f}\left(\sum_i \beta_{if} p_i^{1-\sigma_f}\right)^{\frac{1}{1-\sigma_f}} = \bar{c}_f \left(\sum_i \tilde{\beta}_{if}\left(\frac{p_i}{\bar{p}_i}\right)^{1-\sigma_f}\right)^{\frac{1}{1-\sigma_f}}$$

- Input demand function:

$$b_{if}(\mathbf{p}, g_f) = g_f \cdot \beta_{if} \cdot \psi_f^{\sigma_f - 1}\left(\frac{c_f}{p_i}\right)^{\sigma_f} = g_f \cdot \tilde{\beta}_{if} \cdot \left(\frac{c_f}{p_i}\right)^{\sigma_f}\left(\frac{\bar{c}_f}{\bar{p}_i}\right)^{1-\sigma_f}$$

Functional forms

- CET unit revenue function

$$r_f(\mathbf{p}) = \frac{1}{\chi_f}\left(\sum_i \mu_{if} p_i^{1+\theta_f}\right)^{\frac{1}{1+\theta_f}} = \bar{r}_f\left(\sum_i \tilde{\mu_{if}}\left(\frac{p_i}{\bar{p}_i}\right)^{1+\theta_f}\right)^{\frac{1}{1+\theta_f}}$$

- Output supply function:

$$y_{if}(\mathbf{p}, g_f) = g_f \cdot \mu_{if} \cdot \chi_f^{-\theta_f - 1}\left(\frac{r_f}{p_i}\right)^{-\theta_f} = g_f \cdot \tilde{\mu_{if}} \cdot \left(\frac{r_f}{p_i}\right)^{-\theta_f}\left(\frac{\bar{r}_f}{\bar{p}_i}\right)^{1+\theta_f}$$

- CES unit expenditure function

$$e_h(\mathbf{p}) = \frac{1}{\varphi_h}\left(\sum_i \gamma_{ih} p_i^{1-\eta_h}\right)^{\frac{1}{1-\eta_h}} = \bar{e}_h\left(\sum_i \tilde{\gamma_{ih}}\left(\frac{p_i}{\bar{p}_i}\right)^{1-\eta_h}\right)^{\frac{1}{1-\eta_h}}$$

- CES consumer demand function

$$d_{ih}(\mathbf{p}, M_h) = \gamma_{ih} \cdot \varphi_h^{\eta_h - 1}\frac{M_h}{e_h}\left(\frac{e_h}{p_i}\right)^{\eta_h} = \tilde{\gamma_{ih}} \cdot \frac{M_h}{e_h}\left(\frac{e_h}{p_i}\right)^{\eta_h}\left(\frac{\bar{e}_h}{\bar{p}_i}\right)^{1-\eta_h}$$

Model formulation (general form)

- Pricing rules: for each f

$$c_f(\mathbf{p}) \geq r_f(\mathbf{p})$$

- Market clearing conditions: for each i

$$\sum_f y_{if}(\mathbf{p}, g_f) + \sum_h E_{ih} \geq \sum_f b_{if}(\mathbf{p}, g_f) + \sum_h d_{ih}(\mathbf{p}, M_h)$$

- Budget constraint: for each h

$$M_h = \sum_i E_{ih} p_i$$

Calibration

- Set values for $\bar{\mathbf{p}}$; $\bar{\mathbf{e}}$; $\bar{\mathbf{r}} = \bar{\mathbf{c}}$
- Take value information from the MCM, calculate benchmark quantities
- Assume elasticity values
- Calibrate endowments
- Set a numeraire: e.g. $\sum_h e_h = \sum_h \bar{e}_h$

MCM with trade flows

		reg 1 x_1	reg 2 x_2	reg 3 x_3	hh. 1 u_1	hh. 2 u_2	hh. 3 u_3	Σ
p_1	goods from 1	10			-3.22	-2.86	-3.92	0
p_2	goods from 2		20		-2.86	-10.17	-6.97	0
p_3	goods from 3			30	-3.92	-6.97	-19.11	0
w_1	factor in 1	-10			10			0
w_2	factor in 2		-20			20		0
w_3	factor in 3			-30			30	0
	Σ	0	0	0	0	0	0	0

- Single industry per region, producing a single good sold in all regions
- Single immobile factor of production per region owned by the local household

Functional forms

- Change of notation: f now counts firms (origins), goods, and factors, h counts consumers (destinations)
- Expenditure and consumer demand functions stay as before
- The technology of the firms is now trivial:

$$c_f(w_f) = \frac{1}{\psi_f} w_f = \frac{\bar{c}_f}{\bar{w}_f} w_f$$
$$b_f(w_f, g_f) = \frac{1}{\psi_f} g_f = \frac{\bar{c}_f}{\bar{w}_f} g_f$$

- Revenue function is also very simple:

$$r_f(p_f) = \frac{1}{\chi_f} p_f = \frac{\bar{r}_f}{\bar{p}_f} p_f$$
$$y_f(p_f, g_f) = \frac{1}{\chi_f} g_f = \frac{\bar{r}_f}{\bar{p}_f} g_f$$

Functional forms

- CES unit expenditure function

$$e_h(\mathbf{p}) = \frac{1}{\varphi_h}\left(\sum_i \gamma_{fh} p_f^{1-\eta_h}\right)^{\frac{1}{1-\eta_h}} = \bar{e}_h\left(\sum_f \tilde{\gamma}_{fh}\left(\frac{p_f}{\bar{p}_f}\right)^{1-\eta_h}\right)^{\frac{1}{1-\eta_h}}$$

- CES consumer demand function

$$d_{fh}(\mathbf{p}, M_h) = \gamma_{fh}\cdot\varphi_h^{\eta_h-1}\frac{M_h}{\tilde{e}_h}\left(\frac{\tilde{e}_h}{p_f}\right)^{\eta_h} = \tilde{\gamma}_{fh}\cdot\frac{M_h}{e_h}\left(\frac{e_h}{p_f}\right)^{\eta_h}\left(\frac{\bar{e}_h}{\bar{p}_f}\right)^{1-\eta_h}$$

Model formulation

- Set values for $\bar{\mathbf{p}}; \bar{\mathbf{w}}; \bar{\mathbf{e}}; \bar{\mathbf{r}} = \bar{\mathbf{c}}$
- Pricing rules: for each f

$$\frac{w_f}{\bar{w}_f} = \frac{p_f}{\bar{p}_f}$$

- Market clearing for goods: for each f

$$\frac{\bar{r}_f}{\bar{p}_f} g_f = \sum_h \tilde{\gamma}_{fh} \cdot \frac{M_h}{e_h} \left(\frac{e_h}{p_f}\right)^{\eta_h} \left(\frac{\bar{e}_h}{\bar{p}_f}\right)^{1-\eta_h}$$

- Market clearing for factors: for each f

$$\sum_h E_{fh} = \frac{\bar{c}_f}{\bar{w}_f} g_f$$

- Income balance:

$$M_h = \sum_f E_{fh} w_f$$

- Set a numeraire: e.g. $\sum_h e_h = \sum_h \bar{e}_h$