The simple 2x2x1 model and extensions

A. Model formulation using the algorithm

(1) Assume the following input-output information about an economy to be given (this is just an example):

	Firm X	Firm Y
Good X	+100	
Good Y		+100
Factor L	-60	-40
Factor K	-40	-60

Assume that factor payments belong to a single household who spends all this income to buy goods X and Y.

We want to set up a GE model that allows studying the impacts of external shocks on relative prices of labour and capital.

(2) When we have so little information, it is easy to define the agents. We have two producing firms and one consumer. The purpose of this step becomes more clear when there is a lot of data and one has to aggregate households, firms, regions etc, if that is necessary in order to simplify the model a bit.

(3) As mentioned earlier, the central point of the CGE approach is an assumption that a certain set of data describes equilibrium of a deterministic model. This point can be illustrated by constructing a “benchmark equilibrium dataset”, which is also a nice pedagogical tool. There are two forms of this in use: a micro-consistent matrix (*MCM*) and a social accounting matrix (SAM). We will now organize the data described above into an MCM. An MCM has agents in columns and markets in rows:

	Firm X	Firm Y	Consumer
Good X	+100		-100
Good Y		+100	-100
Factor L	-60	-40	+100
Factor K	-40	-60	+100

Positive value in the cell mean a sale (receipt) and negative – a purchase (payment) by respective agent. Now, why is this an example of a “benchmark equilibrium dataset”? Note that the rows and columns sum up to zero. This fact illustrates the *general equilibrium consistency* of the underlying economy: zero column sums indicate that the expenditures of the respective agents are consistent with their income, and zero row sums indicate that the value of demand equals the value of supply on respective markets. Thus, we can try to define a model that would have this dataset as its equilibrium outcome.

(4) We start with classical Walrasian setup: perfect competition, price taking from the side of all agents. The data also suggests that firms earn zero profits, and no taxes are collected, so only one set of prices has to be defined.

(5) No institutions defined: no closure rules

(6) There are virtually only two constraints on the functional forms:

- they must be consistent with GE theory (thus, demand functions should be homogenous of degree 0, continuous, nonnegative, satisfy the Walras' law),
- they must be analytically tractable (it should be possible to derive demand functions from the cost functions)

Normally, the modellers are not particularly inventive here and stick to the family of convenient functional forms (Cobb-Douglas, Constant Elasticity of Substitution, Linear Expenditure System, Almost Ideal Demand System) that are known to satisfy these requirements.

Additional constraints regarding the choice of functional forms may come from prior knowledge of production process, e.g. the ease of substitution between factors, or similar considerations for the consumer demand, e.g. subsistence level (minimum consumption needs).

Overall, we have to specify the following functions:

1. Cost function Firm X => Input demand functions Firm X
2. Cost function Firm Y => Input demand functions Firm Y
3. Revenue function Firm X => Output supply functions Firm X
4. Revenue function Firm Y => Output supply functions Firm Y
5. Consumer demand functions

These will be building blocks for:

- market clearing conditions
- firms' optimality conditions
- budget constraint

In this test setup, we just want to make the model run as soon as possible. So, lets assume that the firms have a constant returns to scale Cobb-Douglas (CRS CD) technology and the household has a Cobb-Douglas utility. Note that revenue functions are simply linear in this single output case.

Now, lets write all these functions down. We will try to keep the same notation as on the slides.

A CRS CD cost function:

$$c_X(w_L, w_K, g_X) = g_X w_L^{\alpha_X} w_K^{1-\alpha_X};$$

$$c_Y(w_L, w_K, g_Y) = g_Y w_L^{\alpha_Y} w_K^{1-\alpha_Y}.$$

g-s and *w*-s are endogenous variables, α -s are exogenous share parameters.

Corresponding factor demand functions:

$$b_{XL}(\cdot) = \frac{\alpha_X c_X}{w_L}; \; b_{XK}(\cdot) = \frac{(1-\alpha_X)c_X}{w_K};$$

$$b_{YL}(\cdot) = \frac{\alpha_Y c_Y}{w_L}; \; b_{YK}(\cdot) = \frac{(1-\alpha_Y)c_Y}{w_K}.$$

CRS: throughput quantity generated by the production function is equal to the throughput quantity used by the input requirement function, $z_X(g_X) = g_X$.

Revenue functions and supply functions: trivial case of single output ($h_X(y) = y_X$)

$$y_X^* = g_X \qquad => \qquad r_X(p_X, g_X) = p_X \cdot g_X$$

$$y_Y^* = g_Y \qquad => \qquad r_Y(p_Y, g_Y) = p_Y \cdot g_Y .$$

CRS case, no mark-ups: AC=P, zero profits.

Cobb-Douglas consumer demand functions (note: this is Marshalian demand):

$$d_X(p_X, p_Y, M) = \frac{\alpha_c M}{p_X};$$

$$d_Y(p_X, p_Y, M) = \frac{(1-\alpha_c)M}{p_Y}.$$

Now, write down the complete equilibrium system:

Market clearing conditions:

$$g_X \geq \frac{\alpha_c M}{p_X} \qquad \perp \qquad p_X \geq 0$$

$$g_Y \geq \frac{(1-\alpha_c)M}{p_Y} \qquad \perp \qquad p_Y \geq 0$$

$$E_L \geq \frac{\alpha_X c_X(.)}{w_L} + \frac{\alpha_Y c_Y(.)}{w_L} \qquad \perp \qquad w_L \geq 0$$

$$E_K \geq \frac{(1-\alpha_Y)c_Y(.)}{w_K} + \frac{(1-\alpha_X)c_X(.)}{w_K} \qquad \perp \qquad w_K \geq 0$$

Note that c_X, c_Y are functions of prices and activity levels.

Optimality conditions:

$$w_L^{\alpha_X} w_K^{1-\alpha_X} \geq p_X \qquad \perp \qquad g_X \geq 0$$

$$w_L^{\alpha_Y} w_K^{1-\alpha_Y} \geq p_Y \qquad \perp \qquad g_Y \geq 0$$

Budget constraint:

$$M = w_L E_L + w_K E_K$$

Now we need a numéraire. Recall that we noted that unit expenditure could be used as a perfect price index in a single consumer case. A CD unit expenditure function corresponding to our demand functions is:

$$\tilde{e}(p_X, p_Y) = p_X^{\alpha_c} p_Y^{1-\alpha_c} = \overline{P} .$$

We will set $\overline{P} = const$.

Finally, we need to remember about the Walras' law. It is not possible to solve the model as formulated above for unique values of endogenous variables, as absolute prices are not defined. However, we can drop one market clearing condition and introduce a numéraire definition instead. We will drop the equation for capital market.

This finishes the model formulation

(7) Calibration

Now we need to set the values of exogenous parameters α_X, α_Y, α_C, E_L, E_K, $\overline{P}$. Because they are parameters, we can set them to whatever values (keeping shares between 0 and 1). However, we are constrained by the need to reproduce the data in the MCM. *Calibration* is a process of making the model reproduce the benchmark dataset as a model solution. This notion is tightly linked to the assumption of an initial "observable" equilibrium. In the process of calibration, the equations of the models are "inverted," such that the parameters become the unknowns, and the endogenous variables become parameters with their values taken from the data or freely chosen (e.g. in the case of prices).

Note that, in our notation, the MCM contains the following information:

	Firm X	**Firm Y**	**Consumer**
Good X	$p_X^0 g_X^0$		$-p_X^0 d_X^0$
Good Y		$p_Y^0 g_Y^0$	$-p_Y^0 d_Y^0$
Factor L	$-w_L^0 b_{XL}^0$	$-w_K^0 b_{XK}^0$	$w_L^0 E_L$ (VL^0)
Factor K	$-w_L^0 b_{YL}^0$	$-w_K^0 b_{YK}^0$	$w_K^0 E_K$ (VK^0)

or, substituting the demand functions

	Firm X	**Firm Y**	**Consumer**
Good X	$p_X^0 g_X^0$		$-\alpha_c M^0$
Good Y		$p_Y^0 g_Y^0$	$-(1-\alpha_c) M^0$
Factor L	$-\alpha_X c_X^0$	$-\alpha_Y c_Y^0$	$w_L^0 E_L$ (VL^0)
Factor K	$-(1-\alpha_X) c_X^0$	$-(1-\alpha_Y) c_Y^0$	$w_K^0 E_K$ (VK^0)

The CD shares $\alpha_X = 0.6$, $\alpha_Y = 0.4$, $\alpha_C = 0.5$ are thus easily calculated using the entries in the MCM (in value terms).

To determine E_L and E_K, however, it is not enough to have the value data. We have to make the price and quantity distinction (in other words, *choose units*). A widely used approach is to set the prices equal to 1. If we choose $w_L^0 = w_K^0 = 1$, it will yield $p_X^0 = 1$, $p_Y^0 = 1$, and $\overline{P} = 1$. Then $E_L = E_K = 100$, $g_X^0 = g_Y^0 = 100$.

B. Choice of units

We are free to make other choice of units for the endowment commodities. For example, $w_L^0 = 5$ and $w_K^0 = 20$ (for whatever reason). Then, in order to replicate benchmark values, a more general formula of the price index must be used, namely:

$$\overline{P} = \left(\left(w_L^0 \right)^{\alpha_X} \left(w_K^0 \right)^{1-\alpha_X} \right)^{\alpha_C} \left(\left(w_L^0 \right)^{\alpha_Y} \left(w_K^0 \right)^{1-\alpha_Y} \right)^{1-\alpha_C}$$

We can also choose a completely different numéraire: we can set wage rate to always equal 1, we can use expressions involving sums or products of other prices – in this setting, this will not affect the results of policy experiments.

C1. Introducing taxes

First, we show how to shock the basic model by means of a tax. Then, we will show how to calibrate the model to the MCM already containing tax receipts.

To introduce a possibility for a factor tax shock, we rewrite two of the equations: the pricing rule for good X and the budget constraint. In this simple setup, we assume that there will be a homogenous tax on all inputs for firm X, and that the resulting tax revenue will be simply given to the consumer. In the exercises, you are asked to introduce state as a separate player with its own objectives.

For now, nothing changes in the MCM, but we simply leave place for the new entries that will appear after the shock.

	Firm X	Firm Y	Consumer
Good X	+100		-100
Good Y		+100	-100
Factor L	-60	-40	+100
Factor K	-40	-60	+100
Tax on X	0		0

Now, we change the pricing rule to read:

$$w_L^{\alpha_X} w_K^{1-\alpha_X} (1+t_X) \geq p_X \qquad \perp \qquad g_X \geq 0$$

We can write it simply this way, because both inputs are equally taxed. Tax rate is defined as $0 < t_X < 1$. We introduce it as a parameter and set its initial value to zero.

The budget constraint will now read:

$$M = w_L E_L + w_K E_K + t_X c_X$$

Now we can set the tax rate to an arbitrary value and see what effects this would bring.

C2. Benchmark taxes

Next, we turn to a more complex example with existing labour taxes that might be not equal between the two firms.

The MCM:

	Firm X	Firm Y	Consumer
Good X	+100		-100
Good Y		+100	-100
Factor L	-40	-30	+70
Factor K	-40	-60	+100
Tax on L	-20	-10	+30

With taxes, we have to be careful during calibration, so that we don't confuse prices including taxes and excluding taxes.

Let's choose the convention that the factor prices prior to taxes are equal to 1. Then, the endowments are simply equal to the values in the MCM. The consequence of this approach is however that we cannot set benchmark prices of goods at unity any more. Tax rates are calibrated from the MCM values. Then, the goods prices are calibrated, as well as the initial activity levels. We introduce taxes in all equations where cost functions and their derivatives are involved. Note that the wage is paid at untaxed rate. The collected tax revenue is then given to the consumer.

The *Hicksian equivalent variation* EV_r is defined as an absolute increase in before-shock disposable income that under the before-shock prices would render the after-shock level of utility. We will use the relative equivalent variation (e.g. real income change) measure to evaluate the welfare implication of a given shock:

$$REV = \frac{M^1 / \overline{P}^1}{M^0 / \overline{P}^0} - 1 .$$

Exercises:

In principle, we may introduce a new set of variables that would denote after-tax prices. This would make mathematical notation simpler but would make the system of equations longer (we leave this for an exercise). The introduction of state budget is also left as an exercise.

D. Walras law

This is a conceptually important point that is often neglected in the model descriptions. Checking whether Walras law holds can be thought as a consistency check for the whole model. In order to check this, we introduce an explicit variable, called Walras, into the model. It is complementary with the price-level equation. If somewhere in the model there is an income that has no corresponding expenditure, or vice versa, Walras law will be violated, and we will see it in the value of the corresponding variable.

E. Slack activities

Here we try to model a situation where one product may be produced using two technologies, one more efficient than the other. An inefficient activity is initially not active, but when the efficient activity is highly taxed, it starts operating. By this, we can simulate the way by which some small-scale production in the shadow economy is operating. Of course, the assumption of perfect substitution between the products of efficient and inefficient activity is not too realistic.

	Firm X	***Firm Y***	***Firm Z***	***Consumer***
Good X	+100			-100
Good Y		+100		-100
Factor L	-60	-40		+100
Factor K	-40	-60		+100

As the activity of firm Z is not observable in the initial equilibrium, we have to simply guess its technology. We assume that it is 10% less efficient than firm X:

$$c_Z(w_L, w_K, g_Z) = 1.1 \cdot g_Z w_L^{\alpha_X} w_K^{1-\alpha_X}$$

Optimality conditions:

$$w_L^{\alpha_X} w_K^{1-\alpha_X} \geq p_X \quad \perp \quad g_X \geq 0$$

$$w_L^{\alpha_Y} w_K^{1-\alpha_Y} \geq p_Y \quad \perp \quad g_Y \geq 0$$

$$1.1 \cdot w_L^{\alpha_X} w_K^{1-\alpha_X} \geq p_X \quad \perp \quad g_Z \geq 0$$

Market equilibrium conditions are also changed correspondingly.

Now, if we impose a tax affecting firm X, the “shadow” technology will become a cheaper producer of good X, and the “official” technology with stop producing,