A note on the CES functional form: essential derivations

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1 CES Production Function

$$g = \psi\Big(\sum_i \beta_i^{1-\rho} b_i^\rho\Big)^{1/\rho}, i \in I = [1...N] \tag{1}$$

The corresponding firm's optimization problem (cost minimization) is

$$\min_x \quad \sum_i p_i x_i \quad s.t. \quad g = \psi(\sum_i \beta_i^{1-\rho} b_i^\rho)^{1/\rho}.$$

with first order conditions given by:

$$p_k = \lambda \cdot \psi \cdot \beta_k^{1-\rho} \cdot b_k^{\rho-1} (\sum_i \beta_i^{1-\rho} b_i^\rho)^{\frac{1}{\rho}-1},$$

$$p_j = \lambda \cdot \psi \cdot \beta_j^{1-\rho} \cdot b_j^{\rho-1} (\sum_i \beta_i^{1-\rho} b_i^\rho)^{\frac{1}{\rho}-1},$$

for $j \in I$, $k \in I$, and where λ is the Lagrange multiplier.

Hence

$$\frac{p_k}{p_j} = \frac{\beta_k^{1-\rho} b_k^{\rho-1}}{\beta_j^{1-\rho} b_j^{\rho-1}} \quad \Rightarrow \quad \frac{p_k b_k}{p_j b_j} = \frac{\beta_k^{1-\rho}}{\beta_j^{1-\rho}} \cdot \Big(\frac{b_k}{b_j}\Big)^\rho;$$

which gives the intuition of $\beta_j^{1-\rho}$-s as coefficients determining the *functional* distribution of income.

$$b_j^{\rho-1} = \frac{\beta_k^{1-\rho} p_j}{\beta_j^{1-\rho} p_k} \cdot b_k^{\rho-1} \quad \Rightarrow \quad b_j^\rho = \Big(\frac{\beta_k^{1-\rho} p_j}{\beta_j^{1-\rho} p_k}\Big)^{\frac{\rho}{\rho-1}} \cdot b_k^\rho;$$

Substituting into the production function we get

$$g = \psi\Bigg(\sum_i \beta_i^{1-\rho} b_k^\rho \cdot \Big(\frac{\beta_k^{1-\rho} p_i}{\beta_i^{1-\rho} p_k}\Big)^{\frac{\rho}{\rho-1}}\Bigg)^{1/\rho}.$$

Solving for factor demand will give

$$b_k = \frac{g}{\psi} \cdot \Big(\frac{\beta_k^{1-\rho}}{p_k}\Big)^{\frac{1}{1-\rho}} \Big(\sum_i \beta_i p_i^{\frac{\rho}{\rho-1}}\Big)^{-1/\rho}$$

Now, using the fact that the elasticity of factor substitution in (1) is equal to $\frac{1}{1-\rho}$ and defining this elasticity as σ, we can rewrite the last expression as follows:

$$b_k = \frac{g}{\psi} \cdot \Big(\frac{\beta_k^{1/\sigma}}{p_k}\Big)^\sigma \Big(\sum_i \beta_i p_i^{1-\sigma}\Big)^{\frac{\sigma}{1-\sigma}} \tag{2}$$

Finally, the expression for the unit cost $c = \frac{1}{y} \sum_i p_i b_i$ is

$$c = \frac{1}{\psi} \cdot \frac{\sum_i \beta_i p_i^{1-\sigma}}{\Big(\sum_i \beta_i p_i^{1-\sigma}\Big)^{\frac{-\sigma}{1-\sigma}}} = \frac{1}{\psi} \cdot \Big(\sum_i \beta_i p_i^{1-\sigma}\Big)^{\frac{1}{1-\sigma}}, \tag{3}$$

which allows us to rewrite (2) as:

$$b_k = g \cdot \beta_k \cdot \psi^{\sigma-1} \cdot \Big(\frac{c}{p_k}\Big)^\sigma.$$

2 CES Utility Function

$$U = \varphi\Big(\sum_i \gamma_i^{1-\varsigma} d_i^{\varsigma}\Big)^{1/\varsigma} \tag{4}$$

The corresponding individual's optimization problem is

$$\max_x \quad \varphi(\sum_i \gamma_i^{1-\varsigma} d_i^{\varsigma})^{1/\varsigma} \quad s.t. \quad \sum_i p_i d_i = M.$$

Using the first order conditions analogously to previous section produces

$$\frac{p_k}{p_j} = \frac{\gamma_k^{1-\varsigma} d_k^{\varsigma-1}}{\gamma_j^{1-\varsigma} d_j^{\varsigma-1}}$$

$$d_j = \Big(\frac{\gamma_k^{1-\varsigma} p_j}{\gamma_j^{1-\varsigma} p_k}\Big)^{\frac{1}{\varsigma-1}} \cdot d_k.$$

Now summing up:

$$M = \sum_i p_i d_i = d_k \cdot \Big(\frac{\gamma_k^{1-\varsigma}}{p_k}\Big)^{\frac{1}{1-\varsigma}} \Big(\sum_i p_i^{\frac{\varsigma}{\varsigma-1}} \gamma_i\Big)$$

Inverting the expression and switching to syntax in terms of $\eta = 1/(1-\varsigma)$ finally gives the formula for consumer demand:

$$d_k = \Big(\frac{\gamma_k^{1/\eta}}{p_k}\Big)^{\eta} \cdot \frac{M}{\sum_i \gamma_i p_i^{1-\eta}} \tag{5}$$

This can be inserted into (4) to produce the expression for indirect utility:

$$V = M \cdot \varphi\Big(\sum_i \gamma_i p_i^{1-\eta}\Big)^{\frac{1}{\eta-1}}$$

Employing duality principles, the expenditure function is:

$$E = U_0 \cdot \frac{1}{\varphi}\Big(\sum_i \gamma_i p_i^{1-\eta}\Big)^{\frac{1}{1-\eta}}. \tag{6}$$

Denoting unit expenditure E/U_0 by e, (5) becomes:

$$d_k = \gamma_k \cdot \varphi^{\eta-1}\Big(\frac{e}{p_k}\Big)^{\eta} \cdot \frac{M}{e}.$$

3 Calibrated Share Form

For the purposes of model specification, it is often not really necessary to calibrate the values of the parameters $\psi, \beta, \varphi, \gamma$ from above. It is possible to skip this step by applying the following procedure:

1. Write down all equations evaluated at the initial equlibrium
2. Express the shift and share parameters as functions of the initial values of variables
3. Substitute the derived expressions for the parameters in the general formulation of the model

Let $\bar{x}$ denote the benchmark value of a given variable.

For the case of a CES demand function: Let $\tilde{\gamma_k} = \frac{\bar{d_k}\bar{p_k}}{\bar{M}}$ denote the benchmark expenditure share of good i. From (5) evaluated at the benchmark we obtain:

$$\gamma_k = \bar{d_k}\varphi^{1-\eta}\left(\frac{\bar{e}}{\bar{p_k}}\right)^{-\eta}\frac{\bar{e}}{\bar{M}} = \varphi^{1-\eta}\tilde{\gamma_k}\left(\frac{\bar{e}}{\bar{p_k}}\right)^{1-\eta} \tag{7}$$

Now, substituting (7) into (6), we get

$$e = \frac{1}{\varphi}\left(\sum_i \tilde{\gamma_i}\varphi^{1-\eta}\left(\frac{\bar{e}}{\bar{p_i}}\right)^{1-\eta} p_i^{1-\eta}\right)^{\frac{1}{1-\eta}} = \bar{e}\left(\sum_i \tilde{\gamma_i}\left(\frac{\bar{p_i}}{\bar{p_i}}\right)^{1-\eta}\right)^{\frac{1}{1-\eta}}$$

Further, substituting (7) into (5), we get

$$d_k = \tilde{\gamma_i}\left(\frac{\bar{e}}{\bar{p_k}}\right)^{1-\eta}\left(\frac{e}{p_k}\right)^{\eta}\frac{M}{e} = \bar{d_k}\left(\frac{e}{\bar{e}}\right)^{\eta-1}\left(\frac{p_k}{\bar{p_k}}\right)^{-\eta}\frac{M}{\bar{M}}.$$

In the same manner, we can derive:

$$c = \bar{c}\left(\sum_i \tilde{\beta_i}\left(\frac{\bar{p_i}}{\bar{p_i}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}},$$

$$b_k = \tilde{\beta_k}\left(\frac{\bar{c}}{\bar{p_k}}\right)^{1-\sigma}\left(\frac{c}{p_k}\right)^{\sigma} g = \bar{b_k}\left(\frac{c\cdot\bar{p_k}}{\bar{c}\cdot p_k}\right)^{\sigma}\frac{g}{\bar{g}},$$

where $\tilde{\beta_i} = \frac{\bar{b_i}\cdot\bar{p_i}}{\bar{g}\cdot\bar{c}}$